## Miscellaneous exercise 7

1 Gin is 45% alcohol by volume and tonic is non-alcoholic. It is required to mix gin-and-tonics which are at least 10% alcohol by volume and contain at least 200 ml of liquid but at most 30 ml of alcohol.

(a) What are the least and greatest amounts of gin which such a drink could contain?

(b) What are the least and the greatest amounts of tonic which such a drink could contain?

2 Solve the two linear programming problems:

(a) Maximise $2x + 3y$,
subject to $5x + 7y \leqslant 35$,
$4x + 9y \leqslant 36$,
$x \geqslant 0, y \geqslant 0$.

(b) Maximise $2x + 3y$,
subject to $5x + 7y \leqslant 35$,
$4x + 9y \leqslant 36$,
$x \geqslant 0, y \geqslant 0$,
$x, y$ are integers.

3 The following figures can be used to compare the three main forms of investment.

| | Deposits | Gilts | Equities |
|---|---|---|---|
| Expected return[1] | 1.5% | 2.5% | 8% |
| Risk factor[2] | 0 | 20 | 60 |

[1] Real return is inflation adjusted. Based upon the average return over the last 80 years. (Source: Barclays Capital Equity Gilt Study 1999)
[2] Based upon the expected volatility of the invested capital.

A retired person has £70,000 to invest to supplement their pension. A stockbroker recommends that they should have a mix of investments to give an expected return of at least 5.5% and an overall risk factor of at most 40. Subject to these constraints, the pensioner wishes to maximise the amount of money in fixed interest investment (gilts).

(a) Let £$x$, £$y$ and £$z$ be the amounts invested in deposits, gilts and equities respectively. Show that the problem can be formulated as follows.

Maximise $y$,
subject to $x + y + z = 70\,000$, $8x + 6y \leqslant 5z$,
$2x + y \geqslant z$, $x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$.

(b) What main assumption has been made in this formulation of an investment problem?

(c) Solve the linear programming problem of part (a). (Hint: substitute for $z$.)

4 (a) Sketch the feasible region for the following constraints.

$-7x + 10y \leqslant 70$, $x + y \leqslant 15$, $x \leqslant 7$,
$7x - 4y \leqslant 28$, $21x + 7y \geqslant 63$, $5x + 11y \geqslant 55$,
$x, y \geqslant 0$.

(b) What value of $x$ maximises

(i) $4x + 5y$, (ii) $5x + 4y$?

5 A university has enough places for 5000 students. Government restrictions mean that at least 80% of the places must be given to UK students, but the remainder may be given to overseas students.

There are 2000 residential places available in the halls. All overseas students and at least one-third of the UK students must be given places in the halls.

The university gets £4000 in tuition fees for each UK student and £6000 for each overseas student. It wants to maximise the fees received.

Let $x$ be the number of places given to UK students and $y$ be the number of places given to overseas students.

(a) Explain, briefly, why the problem requires the function $P = 4000x + 6000y$ to be maximised.

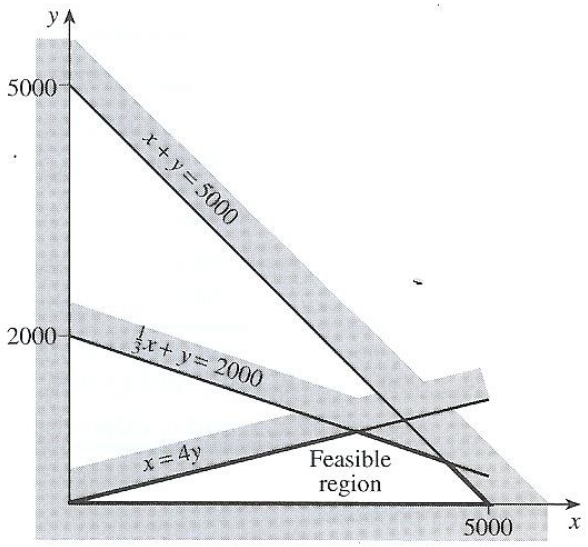
(b) The constraints of the problem are

$$x + y \leqslant 5000,$$
$$4y \leqslant x,$$
$$\tfrac{1}{3}x + y \leqslant 2000,$$
$$x \geqslant 0 \text{ and } y \geqslant 0.$$

Explain why

(i) $4y \leqslant x$, and (ii) $\tfrac{1}{3}x + y \leqslant 2000$.

The feasible region for the problem is shown in the figure.




(c) Write down the vertices of the feasible region which lie on the $x$-axis. Use simultaneous equations to calculate the coordinates of the other vertices of the feasible region (to the nearest whole number), and hence calculate the value of $P$ at each vertex.

(d) Use your answer to (c) to advise the university on how many places they should give to UK students, and how many to overseas students, to maximise the fees received. (OCR)

6 A company manufactures two kinds of robot, the Brainy and the Superbrainy. After production, each robot is tested for its coordination and its logic. The company wants to maximise its profit from the sale of the robots. You may assume that every robot manufactured is sold. The table below shows the times required by these tests, the time available each week, and the profit per robot.

| | Coordination test (hours) | Logic test (hours) | Profit per robot (£) |
|---|---|---|---|
| Brainy | 6 | 3 | 2400 |
| Superbrainy | 3 | 4 | 3000 |
| Time available (hours) | 60 | 60 | |

This problem is being modelled using a linear programming formulation:

$x$ = number of Brainy robots manufactured per week;

$y$ = number of Superbrainy robots manufactured per week.

Maximise $P = 2400x + 3000y$.

(a) Write down the four constraints for this problem.

(b) Represent the constraints graphically, marking the feasible region clearly.

(c) Showing your method clearly, solve the linear programming problem. (OCR)

7 Lou Zitt has a budget of £2000 to spend on storage units for his office. The storage units must not cover more than 50 $m^2$ of floor space. Lou wants to maximise the storage capacity. The three types of storage unit that he can choose from are shown below.

| Type | Storage capacity ($m^3$) | Floor space covered ($m^2$) | Cost (£) |
|---|---|---|---|
| Antique pine units | 2 | 1 | 100 |
| Beech wood units | 9 | 4 | 500 |
| Cedar wood units | 5 | 3 | 200 |

Suppose that Lou buys $a$ antique pine units, $b$ beech wood units and $c$ cedar wood units.

(a) Write down two constraints that must be satisfied by $a$, $b$ and $c$, other than $a \geqslant 0$, $b \geqslant 0$ and $c \geqslant 0$.

(b) Write down the objective function for this problem.

(c) Set up the problem as an LP formulation. ('LP' stands for 'linear programming'.) You are not expected to solve the problem.

(d) Identify which aspect of the original problem has been overlooked in the LP formulation.

Because of the shape of the office, Lou also needs to consider the widths of the units. The antique pine units are 1 metre wide, the beech wood units are 2 metres wide and the cedar wood units are 3 metres wide. The total width of the units cannot exceed 20 metres.

(e) Show how to incorporate this additional restriction into your LP formulation. (OCR)

8 Simon is preparing for his motorcycle test. The test has two components, a practical test and a theory test.

Simon estimates that each hour spent preparing for the practical test and each half-hour spent preparing for the theory test will increase his final test score by 5 points. He wants to score as many points as possible.

Simon also knows that he needs to increase his score on the practical test by at least 10 points; that he cannot spend more than 9 hours preparing for the test (of which no more than 4 hours can be spent on preparing for the practical test), and that he must spend longer on preparing for the theory test than on preparing for the practical test.

(a) Identify the variables for this problem.

(b) Set up the problem as an LP formulation, with an objective function and a set of constraints that must be satisfied. (OCR)

9 The owner of a sweet shop has a stock of 200 acid drops, 240 bubble gums and 240 candy bars. He uses these to make up two types of mixed bag.

Each bag of type $X$ contains 4 acid drops, 6 bubble gums and 3 candy bars.
Each bag of type $Y$ contains 4 acid drops, 3 bubble gums and 6 candy bars.

The profit on each bag of type $X$ sold is 20 pence, and the profit on each bag of type $Y$ sold is 25 pence.

The owner of the sweet shop wants to maximise the profit on the sales of these bags.

(a) Identify the variables for this problem.

(b) Write down three constraints that must be satisfied by the variables, other than that they must be non-negative integers.

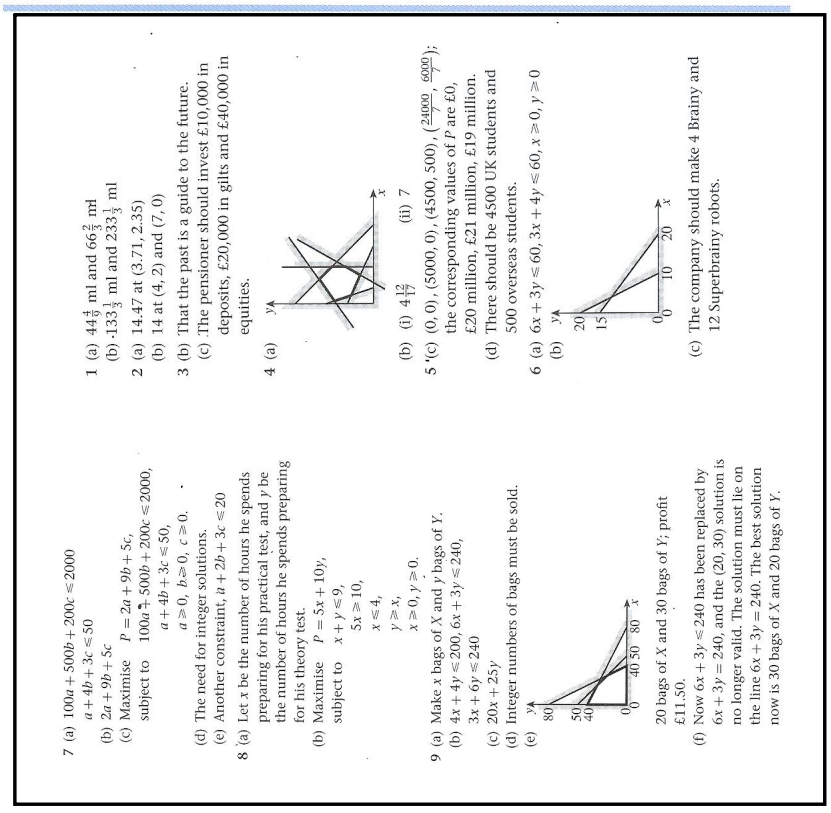
(c) Write down the objective function for this problem.

(d) State an assumption that has been made in writing down the objective function.

(e) Represent the three constraints graphically, and hence find the maximum profit. You should explain your method carefully.

The owner of the sweet shop decides that there should be no bubble gums left over when all the bags have been sold.

(f) Explain how this additional restriction leads to a change in the point that gives the maximum profit, and state the maximum profit in this case. (OCR)



1 (a) $44\frac{4}{9}$ ml and $66\frac{2}{3}$ ml
(b) $133\frac{1}{3}$ ml and $233\frac{1}{3}$ ml

2 (a) 14.47 at (3.71, 2.35)
(b) 14 at (4, 2) and (7, 0)

3 (b) That the past is a guide to the future.
(c) The pensioner should invest £10,000 in deposits, £20,000 in gilts and £40,000 in equities.

4 (a)

(b) (i) $4\frac{12}{17}$ (ii) 7

5 (c) (0, 0), (5000, 0), (4500, 500), $(\frac{24000}{7}, \frac{6000}{7})$; the corresponding values of $P$ are £0, £20 million, £21 million, £19 million.
(d) There should be 4500 UK students and 500 overseas students.

6 (a) $6x + 3y \leqslant 60$, $3x + 4y \leqslant 60$, $x \geqslant 0$, $y \geqslant 0$
(b)

(c) The company should make 4 Brainy and 12 Superbrainy robots.

7 (a) $100a + 500b + 200c \leqslant 2000$
$a + 4b + 3c \leqslant 50$
(b) $2a + 9b + 5c$
(c) Maximise $P = 2a + 9b + 5c$,
subject to $100a + 500b + 200c \leqslant 2000$,
$a + 4b + 3c \leqslant 50$,
$a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$.
(d) The need for integer solutions.
(e) Another constraint, $a + 2b + 3c \leqslant 20$

8 (a) Let $x$ be the number of hours he spends preparing for his practical test, and $y$ be the number of hours he spends preparing for his theory test.
(b) Maximise $P = 5x + 10y$,
subject to $x + y \leqslant 9$,
$5x \geqslant 10$,
$x \leqslant 4$,
$y \geqslant x$,
$x \geqslant 0$, $y \geqslant 0$.

9 (a) Make $x$ bags of $X$ and $y$ bags of $Y$.
(b) $4x + 4y \leqslant 200$, $6x + 3y \leqslant 240$, $3x + 6y \leqslant 240$
(c) $20x + 25y$
(d) Integer numbers of bags must be sold.
(e)

20 bags of $X$ and 30 bags of $Y$; profit £11.50.
(f) Now $6x + 3y \leqslant 240$ has been replaced by $6x + 3y = 240$, and the (20, 30) solution is no longer valid. The solution must lie on the line $6x + 3y = 240$. The best solution now is 30 bags of $X$ and 20 bags of $Y$.