Miscellaneous exercise 5

1 A game has pay-off matrix shown in the figure.

Show that this game has a stable solution and find the play-safe strategies for each player.

A \ B	1	2	3	4
1	6	−3	15	−11
2	7	1	9	5
3	−3	0	−5	8

2 The game 'stone-scissors-paper' has pay-off matrix shown.

	St	Sc	P
St	0	1	−1
Sc	−1	0	1
P	1	−1	0

(a) Suppose that the first player chooses stone, scissors and paper with probabilities p, q and $1-p-q$ respectively. Find the expected gains when the second player chooses each of the strategies stone, scissors and paper.

(b) How can the first player guarantee an expected return of 0?

(c) What is the value of the game? Justify your answer.

3 The pay-off matrix for a zero-sum game between two players A and B is $A\overset{B}{\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}}$.

(a) Show that the game does not have a stable solution.

Player A uses the mixed strategy defined by $(0.6, 0.4)$.

(b) Determine the expected pay-off for A if player B:

(i) plays column one;

(ii) plays column two;

(iii) adopts the strategy $(0.5, 0.5)$.

(c) Determine the optimal strategy for A and its expected pay-off. (OCR)

4 Roland and Colleen play a two-person zero-sum game. The table shows the pay-off matrix for the game. The values in the table are the amounts won by Roland.

		Colleen	
		Stick	Twist
Roland	Stick	−1	4
	Twist	3	−2

(a) Find Roland's and Colleen's play-safe strategies, and hence show that this game does not have a stable solution.

(b) (i) State which strategy Roland should choose if he knows that Colleen will choose her play-safe strategy.

(ii) State which strategy Colleen should choose if she knows that Roland will choose his play-safe strategy.

Roland and Colleen play the game a large number of times. Colleen uses random numbers to choose the Stick strategy with probability p.

(c) Show that the expected gain for Roland when he chooses the Stick strategy is given by $4-5p$, and find a similar expression for the expected gain for Roland when he chooses the Twist strategy.

(d) Use a graphical method to find the optimum value of p. (OCR)

5 Robin is playing a computer game in which he has to protect the environment. He chooses an energy source and the computer chooses the weather conditions.

The numbers of points scored by Robin under each of the combinations of energy type and weather conditions are shown in the table.

		Computer		
		Warm	Wet	Windy
Robin	Atomic energy	−6	3	5
	Bio-gas	2	4	6
	Coal	5	1	3

Robin is trying to maximise his points total and the computer tries to stop him.

(a) Explain why Robin should not choose Atomic energy and why the computer should not choose Windy weather.

(b) Find the play-safe strategies for the reduced game for Robin and for the computer, and hence show that this game does not have a stable solution.

Suppose that Robin uses random numbers to choose Bio-gas with probability p and Coal with probability $1-p$.

(c) Show that the expected loss for the computer when it chooses Warm weather is given by $5-3p$, and find an expression for the expected loss when it chooses Wet weather.

(d) Use a graphical method to find the optimum value of p and the corresponding expected gain for Robin. (OCR)

6 Roy and Callum play a two-person zero-sum game. The table shows the pay-off matrix for the game. The values in the table are the amounts won by Roy.

		Callum Strategy A	Callum Strategy B
Roy	Strategy P	–1	1
	Strategy Q	4	–3

(a) Find Roy's and Callum's play-safe strategies, and show that this game does not have a stable solution.

(b) (i) State which strategy Roy should choose if he knows that Callum will always choose his play-safe strategy.

(ii) State which strategy Callum should choose if he knows that Roy will always choose his play-safe strategy.

Suppose that Roy uses random numbers to choose strategy P with probability p.

(c) Show that the expected gain for Callum when he chooses strategy A is given by $5p - 4$, and find a similar expression for the expected gain for Callum when he chooses strategy B.

(d) Use a graphical method to find the optimum value of p and the corresponding expected gain for Roy. (OCR)

7 Rowena and Colin play a two-person zero-sum simultaneous-play game. The table shows the pay-off matrix for the game.

		Colin Strategy X	Colin Strategy Y	Colin Strategy Z
Rowena	Strategy A	4	–1	2
	Strategy B	4	6	3
	Strategy C	1	2	–2

(a) Find Rowena's and Colin's play-safe strategies, and hence show that this game has a stable solution.

(b) Explain what having a stable solution means to the way the game is played.

(c) Explain why Colin will never choose strategy X, and hence reduce the game to give a 2×2 pay-off matrix.

Suppose that Colin uses random numbers to choose strategy Y with probability p.

(d) Show that the expected gain when Rowena chooses strategy A is given by $2 - 3p$ and find a similar expression for the expected gain when Rowena chooses her other strategy.

(e) Use a graphical method to find the optimum value of p and the corresponding expected gain for Colin. (OCR)

8 Richard and Carol play a two-person zero-sum simultaneous-play game. The table shows the pay-off matrix for the game.

		Carol		
		Strategy X	Strategy Y	Strategy Z
Richard	Strategy A	2	3	–2
	Strategy B	–4	–1	–1
	Strategy C	–5	0	1

(a) Explain the meaning of the term zero-sum game.

(b) Find the play-safe strategies for both Richard and Carol, and hence show that this game does not have a stable solution.

(c) Suppose that Richard knows that Carol will use her play-safe strategy. Explain whether or not he should change from his play-safe strategy as found in part (b).

(d) Suppose that Carol knows that Richard will use his play-safe strategy. Explain whether or not she should change from her play-safe strategy as found in part (b). (OCR)

9 Rose is playing a computer game in which she has to defend a planet from aliens. She chooses a defence strategy and the computer chooses an attack strategy.

The number of points scored by Rose with each combination of strategies is shown in the table.

		Computer		
		Fight	Shoot	Track
Rose	Delay	–2	–5	1
	Hide	3	4	6
	Negotiate	5	–1	2

Rose is trying to maximise the number of points that she scores, and the computer is trying to minimise the number of points that Rose scores.

(a) Find the play-safe strategy for Rose and for the computer, and hence show that this game does not have a stable solution.

(b) Explain why Rose will not choose the Delay strategy.

(c) Which strategy will the computer never choose to play?

Suppose that Rose uses random numbers to choose between her two remaining strategies, choosing the Hide strategy with probability p and the Negotiate strategy with probability $1 - p$.

(d) Find expressions for the expected gain for Rose when the computer chooses each of its remaining strategies.

(e) Calculate the value of p for Rose to maximise her guaranteed return. (OCR)

10 Richard is playing a computer battle game in which he chooses a warrior and the computer chooses an opponent. Neither Richard nor the computer knows what the other has chosen.

The probabilities of Richard winning the battle with each combination of warrior and opponent are shown in the table. If Richard does not win, the computer wins. Both Richard and the computer are playing to win.

		Computer		
		Dragon	Elf	Fighter
Richard	Argent	0.4	0.7	0.3
	Bronze	0.5	0.6	0.8
	Crystal	0.2	0.5	0.1

(a) Explain why the computer should never choose Elf.

(b) Which warrior should Richard never choose?

(c) Use the information from parts (a) and (b) to reduce the table to a 2×2 matrix. Find the play-safe strategies for the reduced game for Richard and for the computer.

(d) Richard plays the game many times. What strategy should he use? Explain your answer. (OCR)

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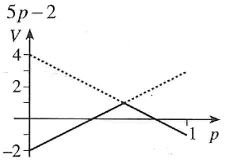
1 $\max(\text{row min}) = \min(\text{col max}) = 1$
Both play option 2.

2 (a) $1-p-2q,\ 2p+q-1,\ q-p$
(b) By choosing $p = q = 1-p-q = \frac{1}{3}$
(c) 0. Each player can guarantee this by choosing their options with equal probabilities.

3 (a) $\max(\text{row min}) = -2$ and $\min(\text{col max}) = 1$; these are unequal so the game does not have a stable solution.
(b) (i) 1 (ii) -0.8 (iii) 0.1
(c) A plays 1 with probability $\frac{3}{8}$; the pay-off is $-\frac{1}{8}$.

4 (a) Roland plays S, gain $\geqslant -1$, and Colleen plays S, loss $\leqslant 3$. As these outcomes are unequal, the game does not have a stable solution.
(b) (i) T (ii) S
(c) $5p-2$
(d)



$\frac{3}{5}$

5 (a) Atomic energy is dominated by Bio-gas, and Windy is dominated by Wet.
(b) $\max(\text{row min}) = 2$ and $\min(\text{col max}) = 4$; these are unequal so the game does not have a stable solution.
(c) $1+3p$
(d) $\frac{2}{3}$, 3

6 (a) P and B; $\max(\text{row min}) = -1$ and $\min(\text{col max}) = 1$; these are unequal so the game does not have a stable solution.
(b) (i) P (ii) A
(c) $3-4p$
(d) $\frac{7}{9}$, $\frac{1}{9}$

7 (a) Rowena: B, guarantees $\geqslant 3$, Colin: Z, guarantees $\leqslant 3$; these are equal, so the game has a stable solution.
(b) There is no advantage to either player in varying from play-safe strategy.
(c) X is dominated by Z;
C is dominated by B *or* A is dominated by B.

	Y	Z
A	−1	2
B	6	3

or

	Y	Z
B	6	3
C	2	−2

(d) $2-3p,\ 3+3p$
(e) 0, −3

8 (a) Zero-sum game means that the gain of one player added to the gain of the other player is zero.
(b) Richard: A, gain $\geqslant -2$;
Carol: Z, loss $\leqslant 1$;
these are unequal so the game does not have a stable solution.
(c) Richard will change to C.
(d) Carol will not change.

9 (a) Rose: Hide, gain $\geqslant 3$;
Computer: Shoot, loss $\leqslant 4$; these are unequal so the game does not have a stable solution.
(b) It is dominated by both of the other strategies.
(c) Track, because it is dominated by Shoot.
(d) Computer chooses Shoot, gain $5p-1$;
computer chooses Fight, gain $5-2p$.
(e) $\frac{6}{7}$

10 (a) Elf is dominated by Dragon.
(b) Crystal
(c) New matrix is

	D	F
A	0.4	0.3
B	0.5	0.8

Richard: Bronze, expected gain 0.5;
Computer: Dragon, expected gain to Richard 0.5.
(d) The game has a stable solution, so Richard should always play Bronze.